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Finite and infinite time ruin probabilities in a stochastic economic environment

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Abstract

Let $(A_1, B_1, L_1), (A_2, B_2, L_2), \ldots$ be a sequence of independent and identically distributed random vectors. For $n \in \mathbb{N}$, denote

$$Y_n = B_1 + A_1 B_2 + A_1 A_2 B_3 + \cdots + A_1 \cdots A_{n-1} B_n + A_1 \cdots A_n L_n.$$

For $M > 0$, define the time of ruin by $T_M = \inf\{n \mid Y_n > M\}$ ($T_M = +\infty$, if $Y_n \leqslant M$ for $n = 1, 2, \ldots$). We are interested in the ruin probabilities for large M. Our objective is to give reasons for the crude estimates $\boldsymbol{P}(T_M \leqslant x \log M) \approx M^{-R(x)}$ and $\boldsymbol{P}(T_M < \infty) \approx M^{-w}$ where $x > 0$ is fixed and $R(x)$ and w are positive parameters. We also prove an asymptotic equivalence $\boldsymbol{P}(T_M < \infty) \sim CM^{-w}$ with a strictly positive constant C. Similar results are obtained in an analogous continuous time model.

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1. Introduction

Let $(A, B, L), (A_1, B_1, L_1), (A_2, B_2, L_2), \ldots$ be independent and identically distributed random vectors. Define the stochastic process $\{Y_n \mid n = 1, 2, \ldots\}$ by

$$Y_n = B_1 + A_1 B_2 + A_1 A_2 B_3 + \cdots + A_1 \cdots A_{n-1} B_n + A_1 \cdots A_n L_n. \tag{1.1}$$

Let M be a positive real number. Define the time of ruin by $T_M = \inf\{n \mid Y_n > M\}$ ($T_M = +\infty$, if $Y_n \leqslant M$ for $n = 1, 2, \ldots$). We are interested in the ruin probabilities for large M. The main emphasis will be in the finite time ruin probabilities but the probability $\boldsymbol{P}(T_M < \infty)$ will also be considered. Additionally, we study the ruin probabilities in an analogous continuous time model. To large extent, they can be analysed in the discrete time framework of model (1.1).

The above process $\{Y_n\}$ and the stopping time T_M are of interest in insurance mathematics. The variables $B_1, B_2, \ldots$ are interpreted as the net payouts and M as the initial capital of an insurance company. The variables $A_1, A_2, \ldots$ are interpreted as the discount factors related to the returns on investments. If $L \equiv 0$ then T_M describes the

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time of ruin, namely, at time T_M, the capital of the company is negative for the first time (if $T_M=+\infty$ then the capital is never negative). See Nyrhinen (1999b) for more details concerning the interpretation. Discounted sums like (1.1) are of interest as such in related fields, for example, in life insurance and finance. We refer to Embrects and Goldie (1994) for a discussion of these and other applications.

We assume throughout the paper that the discount factors are genuine stochastic variables and in particular that $\boldsymbol{P}(A>1)>0$. The interpretation is that there is a risk associated with the investments. We also assume that the variable A is strictly positive. This excludes the possibility of extreme losses related to the investments as explained in Paulsen (1993). For the general background concerning the above processes, we refer to Daykin et al. (1994).

The main objective of the paper is to derive an asymptotic estimate for the finite time ruin probability $\boldsymbol{P}(T_M \leqslant x\log M)$ for given $x>0$. We show that, under suitable conditions, the magnitude of the probability is $M^{-R(x)}$ for large M where $R(x)$ is a specific parameter. The parameter only depends on the distribution of the variable A. A heuristic explanation for this is that the maximum of Y_n, $n\leqslant x\log M$, is in essence determined by the maximum of the products $A_1\cdots A_n$, $n\leqslant x\log M$. Hence, the ruin probability $\boldsymbol{P}(T_M\leqslant x\log M)$ is close to the tail probability

$$\boldsymbol{P}(\max\{A_1\cdots A_n \mid n\leqslant x\log M\}>M). \tag{1.2}$$

This is actually a ruin probability associated with the classical random walk model. Namely, (1.2) equals $\boldsymbol{P}(T'_{\log M}\leqslant x\log M)$ where

$$T'_M=\inf\{n \mid \log A_1+\cdots+\log A_n>M\} \tag{1.3}$$

($T'_M=+\infty$, if $\log A_1+\cdots+\log A_n\leqslant M$ for $n=1,2,\ldots$). See for example Cramér (1955) for the background. All this hints that results of classical ruin theory could be applied to the present model. Indeed, the well known dominating sample path associated with the event $\{T'_{\log M}\leqslant x\log M\}$ gives useful background for our study. See Martin-Löf (1983) for detailed descriptions of such paths. Also the magnitude of probability (1.2) is known to be $M^{-R(x)}$. This follows for example from Corollary 2.5 of Nyrhinen (1998) under suitable conditions. However, the above qualitative views assume appropriate tails for the distributions of the variables A,B and L. See Grey (1994) for the case where the distribution of the variable B is dominating in a similar setup.

Finite time ruin probabilities are much studied in ordinary models where $A\equiv 1$ and $L\equiv 0$. The central limit theorem for the conditional distribution of the time of ruin is derived by several authors with different methods. We refer to Segerdahl (1955), von Bahr (1974) and Siegmund (1975). Large deviations estimates similar to the present paper are obtained in Arfwedson (1955), Martin-Löf (1983,1986) and Nyrhinen (1998). A more detailed description, namely the large deviations principle for the time of ruin is obtained in Collamore (1998) in the multidimensional case and in Nyrhinen (1999a).

In the present model, the earlier studies on the finite time ruin probabilities are mostly directed to non-asymptotic viewpoints. Schnieper (1983) gives a useful recursion associated with the (defective) distribution of T_M. Implicit information is obtained in Paulsen and Gjessing (1997) who find a representation for the Laplace transform of T_M. Norberg (1999a) finds a differential equation for the probabilities in question.

We also pay attention to the infinite time ruin probabilities. Our objective is to show that the magnitude of $\boldsymbol{P}(T_M < \infty)$ is M^{-w} for large M where w is a specific parameter. Also here the parameter is determined by the distribution of the variable A. Additionally, we give sufficient conditions for the asymptotic equivalence

$$\boldsymbol{P}(T_M < \infty) \sim CM^{-w}, \tag{1.4}$$

where C is a positive constant. In Nyrhinen (1999b), we obtained similar results in the case where the processes $\{A_n\}$ and $\{B_n\}$ were independent but general in other respects. Our results here are complementary. We have slightly relaxed the independence assumption but limited the study to the stationary case. We have also incorporated the sequence $\{L_n\}$ into the model. These extensions are useful in the study of the continuous time case in Section 3 below.

There are several other papers which study the probability $\boldsymbol{P}(T_M < \infty)$. Estimate (1.4) and a representation for C are given in Goldie (1991) in the discrete time model (1.1). However, it is not easy to infer directly from the representation whether or not C is positive. This can be done in particular cases, for example, in the case where $\boldsymbol{P}(B > 0) = 1$ and $L \equiv 0$, but these assumptions are rather restrictive. Similar problems appeared in the continuous time model studied in Gjessing and Paulsen (1997). We refer to the discussions after Theorem 6.2 in Goldie (1991) and after Proposition 3.1 in Gjessing and Paulsen (1997) for more details. Our results here give partial solutions to these problems. Based on joint results with Kalashnikov, Norberg (1999b) gives the qualitative view that the probability $\boldsymbol{P}(T_M < \infty)$ cannot be less than a power function of M. A different viewpoint to the problem is taken in Paulsen (1998) where sharp conditions for certain ruin are derived.

The rest of the paper is organized as follows. The main results concerning the ruin probabilities are stated in Section 2. A connection with the continuous time case is presented in Section 3. Section 4 consists of the proofs.

2. Asymptotic estimates for ruin probabilities

Let $(\Omega, S, \boldsymbol{P})$ be a probability space and

$$(A, B, L),\ (A_1, B_1, L_1),\ (A_2, B_2, L_2), \ldots \tag{2.1}$$

independent and identically distributed random vectors on the measurable space (Ω, S). Let the process $\{Y_n \mid n = 1, 2, \ldots\}$ be as in (1.1). For $M > 0$, define the time of ruin T_M by

$$T_M = \begin{cases} \inf\{n \mid Y_n > M\} \\ +\infty \quad \text{if } Y_n \leqslant M \text{ for } n = 1, 2, \ldots . \end{cases} \tag{2.2}$$

We derive in this section asymptotic estimates for the finite and infinite time ruin probabilities. Under certain circumstances, the ruin probabilities equal zero for large M. A criteria for this case is needed and is given and discussed at the end of the section.

The conditions for the results are mainly stated in terms of the generating functions. We refer to Barndorff-Nielsen (1978) for the background. For necessary information concerning large deviations theory, we refer to Dembo and Zeitouni (1993).

We assume that $\boldsymbol{P}(A>0)=1$. Let c be the cumulant generating function of $\log A$ i.e.

$$c(t)=\log \boldsymbol{E}\{\mathrm{e}^{t\log A}\}=\log \boldsymbol{E}\{A^t\} \tag{2.3}$$

for $t\in\mathbb{R}$. Write

$$\mathscr{D}=\{t\in\mathbb{R}\,|\,c(t)<\infty\}. \tag{2.4}$$

Then $\mathscr{D}$ is a convex subset of $\mathbb{R}$ and $0\in\mathscr{D}$. Let $\mathring{\mathscr{D}}$ be the interior of $\mathscr{D}$. Denote

$$w=\sup\{t\,|\,c(t)\leqslant 0\}\in[0,\infty] \tag{2.5}$$

and

$$t_0=\sup\{t\in\mathbb{R}\,|\,c(t)<\infty,\ \boldsymbol{E}\{|B|^t\}<\infty,\ \boldsymbol{E}\{(AL\mathbf{1}(L>0))^t\}<\infty\}\in[0,\infty]. \tag{2.6}$$

Lemma 1. *Assume that $\boldsymbol{P}(A>0)=1$ and that $0<w<t_0\leqslant\infty$. Then, the function c is strictly convex and continuously differentiable on $\mathring{\mathscr{D}}$. Further, $\boldsymbol{P}(A>1)>0$ and $c'(w)>0$.*

We assume the conditions of Lemma 1 in the rest of the section. Denote

$$\mu=1/c'(w)\in(0,\infty) \tag{2.7}$$

and

$$x_0=\lim_{t\to t_0-}(1/c'(t))\in[0,\infty). \tag{2.8}$$

Clearly, $x_0<\mu$.

Let c^* be the Fenchel–Legendre transform of c. By definition, $c^*(v)=\sup\{tv-c(t)\,|\,t\in\mathbb{R}\}$ for $v\in\mathbb{R}$. Define the function $R:(x_0,\infty)\to\mathbb{R}\cup\{\pm\infty\}$ by

$$R(x)=\begin{cases} xc^*(1/x) & \text{for } x\in(x_0,\mu),\\ w & \text{for } x\geqslant\mu. \end{cases} \tag{2.9}$$

This function has been analysed in ruin theory concerning the classical and related models. In particular, it is known by Martin-Löf (1983,1986) and Nyrhinen (1998) that R is finite and continuous on (x_0,∞) and strictly decreasing on (x_0,μ).

We next state the main result of the section. Denote

$$\bar{Y}=\sup\{Y_n\,|\,n=1,2,\ldots\} \tag{2.10}$$

and

$$\bar{y}=\sup\{y\in\mathbb{R}\,|\,\boldsymbol{P}(\bar{Y}>y)>0\}\in(-\infty,\infty]. \tag{2.11}$$

Theorem 2. *Assume the conditions of Lemma* 1 *and that* $\bar{y}=\infty$. *Then,*

$$\lim_{M\to\infty}(\log M)^{-1}\log \boldsymbol{P}(T_M\leqslant x\log M)=-R(x) \tag{2.12}$$

for every $x>x_0$ *and*

$$\lim_{M\to\infty}(\log M)^{-1}\log \boldsymbol{P}(T_M<\infty)=-w. \tag{2.13}$$

Limit (2.12) can be generalized in the following way. Let $I\subseteq[0,\infty)$ be a proper interval such that $I\cap(x_0,\mu)\neq\phi$. Then

$$\lim_{M\to\infty}(\log M)^{-1}\log \boldsymbol{P}(T_M/\log M\in I)=-\inf\{R(v)\,|\,v\in I\cap(x_0,\mu)\}. \tag{2.14}$$

Restricted to (x_0,μ), R may be seen as the large deviations rate function associated with the ruin probabilities. Namely, (2.14) justifies the crude estimate

$$\boldsymbol{P}(T_M/\log M\approx x)\approx M^{-R(x)}$$

for $x\in(x_0,\mu)$. See Nyrhinen (1998,1999a) for further background for this viewpoint.

To see that (2.14) is true, consider as an example an interval $I=(y,x]$ such that $I\subseteq(x_0,\mu)$. Then

$$\boldsymbol{P}(T_M/\log M\in I)=\boldsymbol{P}(T_M\leqslant x\log M)-\boldsymbol{P}(T_M\leqslant y\log M). \tag{2.15}$$

Since R is strictly decreasing on (x_0,μ) we have by Theorem 2

$$\lim_{M\to\infty}\boldsymbol{P}(T_M\leqslant y\log M)/\boldsymbol{P}(T_M\leqslant x\log M)=0. \tag{2.16}$$

Thus the probabilities $\boldsymbol{P}(T_M/\log M\in I)$ and $\boldsymbol{P}(T_M\leqslant x\log M)$ are asymptotically equivalent. We obtain (2.14) by Theorem 2 since by the properties of the function R mentioned above, the right-hand side of (2.14) equals $-R(x)$ in this case.

A refinement of (2.13) can be deduced if additionally, some of the convolution powers of the distribution of $\log A$ has a non-trivial absolutely continuous component. Then

$$M^w\boldsymbol{P}(T_M<\infty)=C+\mathrm{o}(M^{-\varepsilon}) \tag{2.17}$$

when M tends to infinity where C and ε are positive constants. This can be seen by firstly observing that

$$\begin{aligned}\bar{Y}&=\sup\{B_1+A_1L_1,\,B_1+A_1B_2+A_1A_2L_2,\ldots\}\\&=B_1+A_1\sup\{L_1,\ B_2+A_2L_2,\,B_2+A_2B_3+A_2A_3L_3,\ldots\}.\end{aligned} \tag{2.18}$$

It is seen that $\bar{Y}$ satisfies the random equation

$$\bar{Y}=_L B+A\max(L,\bar{Y}), \tag{2.19}$$

where $=_L$ means equality of probability laws. The variables A,B and L are independent of $\bar{Y}$ on the right-hand side of (2.19). Obviously,

$$\boldsymbol{P}(T_M<\infty)=\boldsymbol{P}(\bar{Y}>M) \tag{2.20}$$

for $M>0$. We conclude by (2.13) that $\bar{Y}$ is finite almost surely. Estimate (2.17) is then a consequence of (2.13) and Theorem 6.3 of Goldie (1991). We refer to Section 3 in Nyrhinen (1999b) for more details.

It is natural to assume in Theorem 2 that $\bar{y}$ equals ∞ since otherwise, by (2.20), the ruin probabilities would equal zero for large M. However, the verification of the condition for a given vector (A,B,L) seems not to be easy in general. We next turn to this problem.

For $n \in \mathbb{N}$, denote

$$Y_n^0 = B_1 + A_1B_2 + A_1A_2B_3 + \cdots + A_1 \cdots A_{n-1}B_n \tag{2.21}$$

and

$$\Pi_n = A_1 \cdots A_n. \tag{2.22}$$

Theorem 3. *Assume the conditions of Lemma* 1. *Then* $\bar{y} = \infty$ *if and only if there exists* $k \geqslant 1$ *such that*

$$\boldsymbol{P}(B + AL + Y_k^0/(\Pi_k - 1) > 0,\ \Pi_k > 1) > 0. \tag{2.23}$$

The following examples illustrate the criteria of Theorem 3. In Example 1, we derive rather general sufficient conditions for $\bar{y}$ to equal ∞. The second example is complementary by showing that $\bar{y}$ may equal ∞ beyond the conditions of the first example.

Example 1. Assume the conditions of Lemma 1 and that $\boldsymbol{P}(B + AL \geqslant 0) > 0$ and $\boldsymbol{P}(A > 1, B > 0) > 0$. The variable $B + AL$ in (2.23) is independent of Y_k^0 and Π_k. Thus, condition (2.23) is satisfied for $k = 1$ and so $\bar{y} = \infty$.

Example 2. Let $L \equiv 0$, $\boldsymbol{P}(A = (1+a)^2, B = -1) = p$ and $\boldsymbol{P}(A = 1/(1+a), B = 1) = 1 - p$ with $a > 0$ and $p \in (0,1)$. The conditions of Lemma 1 are satisfied for an appropriate choice of the probability p. Clearly, $\boldsymbol{P}(A > 1,\ B > 0)=0$ and so the result of Example 1 cannot be applied here. Condition (2.23) is satisfied for $k = 2$ since the event

$$\{B = 1,\ A_1 = (1+a)^2,\ B_1 = -1,\ A_2 = 1/(1+a), B_2 = 1\} \tag{2.24}$$

has a positive probability and on that event we have $\Pi_2 = 1 + a > 1$ and

$$B + Y_2^0/(\Pi_2 - 1) = 3 + a > 0. \tag{2.25}$$

Thus $\bar{y} = \infty$ also in this case.

3. The continuous time case

We consider in this section continuous time analogues to the results of Section 2. The idea is to compare the continuous time process with an appropriate discrete time process. We also give an example where our conditions can be verified.

The model to be studied is in essence described in Paulsen (1993). From the viewpoint of ruin theory, the first part of Section 3 of the paper shows connections with our setup here. For the background concerning stochastic integrals and Lévy processes, we refer to Protter (1990) and to Bertoin (1996).

We assume that all the processes below are defined on a filtered probability space $(\Omega, S, \boldsymbol{S}, \boldsymbol{P})$ satisfying the usual conditions. See Protter (1990) for details. Let $\{A_\tau^c \mid \tau \geqslant 0\}$

and $\{B^c_\tau \mid \tau \geqslant 0\}$ be suitable adapted processes on the space $(\Omega, S, \boldsymbol{S}, \boldsymbol{P})$. Specifically, we assume that the process $\{(\log A^c_\tau,\ B^c_\tau) \mid \tau \geqslant 0\}$ has stationary independent increments and that the sample paths of the process are càdlàg (i.e. they are right-continuous and have left limits). Hence, it is a two-dimensional Lévy process. Define the process $\{Y^c_\tau \mid \tau \geqslant 0\}$ by $Y^c_0 = 0$ and

$$Y^c_\tau = \int_0^\tau A^c_{s-} dB^c_s \tag{3.1}$$

for $\tau > 0$.

To illustrate the connection of $\{Y^c_\tau\}$ with the discrete time case, let Y_n be as in (1.1) with $L_n \equiv 0$. Choose

$$A^c_s = \prod_{m=1}^{\lfloor s \rfloor} A_m$$

and

$$B^c_s = \sum_{m=1}^{\lfloor s \rfloor} B_m$$

for $s > 0$ where $\lfloor s \rfloor$ denotes the integer part of s. We then have $Y^c_\tau = Y_n$ for every $\tau \in [n, n+1)$ and $n \in \mathbb{N}$ and so in essence, $\{Y^c_\tau\}$ and $\{Y_n\}$ can be identified. Restricted to the integer time points, the increments of the process $\{(\log A^c_\tau,\ B^c_\tau)\}$ are also stationary and independent. Thus, the process $\{Y^c_\tau\}$ may be seen as a continuous time analogue to $\{Y_n\}$ i.e. we just allow in (3.1) the underlying processes to fluctuate continuously. Observe that A^c_τ is interpreted as the accumulated discount factor and B^c_τ as the accumulated net payout upto time τ while A_n and B_n are interpreted as annual quantities.

For $M > 0$, define the time of ruin T^c_M by

$$T^c_M = \begin{cases} \inf\{\tau \geqslant 0 \mid Y^c_\tau > M\} \\ +\infty \quad \text{if } Y^c_\tau \leqslant M \text{ for every } \tau \geqslant 0. \end{cases} \tag{3.2}$$

We associate with the process $\{Y^c_\tau \mid \tau \geqslant 0\}$ a discrete time process $\{Y_n \mid n = 1, 2, \ldots\}$, namely, we take

$$(A, B, L) =_L (A^c_1,\ Y^c_1,\ (A^c_1)^{-1} \sup\{Y^c_\tau - Y^c_1 \mid \tau \in [0, 1]\}) \tag{3.3}$$

and define $\{Y_n\}$ by (1.1). Observe that the components of the vector (A, B, L) are generally dependent even in the case where the processes $\{A^c_\tau\}$ and $\{B^c_\tau\}$ are independent.

Theorem 4. *Let the process $\{Y^c_\tau \mid \tau \geqslant 0\}$ and the associated discrete time process $\{Y_n \mid n = 1, 2, \ldots\}$ be as described above. Concerning the discrete time process, let the time of ruin T_M, the function R and the parameters w, t_0, μ, x_0 and $\bar{y}$ be as in Section 2. Assume the conditions of Lemma 1. Then, for every $M > 0$ and $n \in \mathbb{N}$,*

$$\boldsymbol{P}(T^c_M \leqslant n) = \boldsymbol{P}(T_M \leqslant n) \tag{3.4}$$

and

$$\boldsymbol{P}(T^c_M < \infty) = \boldsymbol{P}(T_M < \infty). \tag{3.5}$$

By Theorem 4, ruin probabilities in the original model (3.1) can be analysed by means of the associated discrete time model. Gjessing and Paulsen (1997) find similar connections between the discrete and continuous time cases. They also conjecture that, under suitable conditions, $\boldsymbol{P}(T_M^c < \infty) \geqslant \underline{C} M^{-w}$ for large M for some $\underline{C} > 0$. The following corollary supports the conjecture.

Corollary 5. *Assume the conditions of Theorem 4 and that $\bar{y} = \infty$. Then,*

$$\lim_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M^c \leqslant x \log M) = -R(x) \tag{3.6}$$

for every $x > x_0$ and

$$\lim_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M^c < \infty) = -w. \tag{3.7}$$

Assume further that some of the convolution powers of the distribution of $\log A$ has a non-trivial absolutely continuous component. Then there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$M^w \boldsymbol{P}(T_M^c < \infty) = C + \mathrm{o}(M^{-\varepsilon}) \tag{3.8}$$

when M tends to infinity.

To apply Corollary 5, it is necessary to deal with the complicated random vector (3.3). The following example illustrates this part.

Example. Assume that the processes $\{A_\tau^c\}$ and $\{B_\tau^c\}$ are independent and that

$$A_\tau^c = \mathrm{e}^{r\tau + \sigma W_\tau} \tag{3.9}$$

and

$$B_\tau^c = p\tau + X_\tau \tag{3.10}$$

for $\tau \geqslant 0$ where $\{W_\tau \mid \tau \geqslant 0\}$ is standard Brownian motion with $W_0 = 0$, $\{X_\tau \mid \tau \geqslant 0\}$ is a compound Poisson process with $X_0 = 0$ and p, r and σ are real numbers. We assume that $r < 0$ and $\sigma > 0$. Finally, assume that $\boldsymbol{P}(X_1 \geqslant 0) = 1$ and $\boldsymbol{P}(X_1 > 0) > 0$ and that $\boldsymbol{E}\{X_1^t\}$ is finite for some $t > -2r/\sigma^2$.

We verify all the conditions of Corollary 5. Thus, both (3.6) and (3.8) hold. Consider first the requirements concerning the variable A. Clearly, $\boldsymbol{P}(A > 0) = 1$ and

$$c(t) = rt + \sigma^2 t^2/2 < \infty \tag{3.11}$$

for every $t > 0$. Thus $w = -2r/\sigma^2 \in (0, \infty)$. Finally, the distribution of $\log A$ is absolutely continuous.

Consider the requirement $t_0 > w$ of Corollary 5. Observe that (3.1) is now a Lebesgue–Stieltjes integral for every $\omega \in \Omega$. Since r is negative and the sample paths of $\{X_\tau\}$ are increasing we have for $\tau \in [0, 1]$,

$$|Y_\tau^c| = \left| p \int_0^\tau A_{s-}^c \, \mathrm{d}s + \int_0^\tau A_{s-}^c \, \mathrm{d}X_s \right| \leqslant \mathrm{e}^{\sigma \bar{W}} (|p| + X_1), \tag{3.12}$$

where

$$\bar{W} = \sup\{W_s \mid s \in [0, 1]\}. \tag{3.13}$$

It is well known that $\boldsymbol{E}\{e^{t\bar{W}}\}$ is finite for every $t \in \mathbb{R}$. By our assumptions, $\boldsymbol{E}\{X_1^t\}$ is finite for some $t > w$. By (3.12),

$$\begin{aligned}\boldsymbol{E}\{(AL\mathbf{1}(L>0))^t\} &= \boldsymbol{E}\{(\sup\{Y_\tau^c - Y_1^c \mid \tau \in [0,1]\})^t\}\\ &\leqslant \boldsymbol{E}\{(2e^{\sigma\bar{W}}(|p| + X_1))^t\} < \infty \end{aligned} \tag{3.14}$$

for some $t > w$. Similarly, by (3.12), $\boldsymbol{E}\{|B|^t\}$ is finite for some $t > w$. Thus $t_0 > w$.

It remains to check that $\bar{y} = \infty$. We make use of Example 1 of Section 2. By our assumptions, $\boldsymbol{P}(X_1 > 0) > 0$. Thus $\{X_\tau\}$ has strictly positive jumps with a positive probability. The jump sizes are mutually independent and they are also independent of the number of jumps. Since the number of jumps upto time 1 is not bounded we conclude that $\boldsymbol{P}(X_1 > y) > 0$ for every $y \in \mathbb{R}$. Let $\varepsilon \in (0,1)$ be small and

$$\Omega_1 = \{\omega \in \Omega \mid \varepsilon \leqslant A_s^c \leqslant 1/\varepsilon \text{ for every } s \in [0,1],\ A_1^c > 1,\ X_1 \geqslant |p|/\varepsilon^2 + 1\}. \tag{3.15}$$

Then $\boldsymbol{P}(\Omega_1) > 0$ and

$$Y_1^c \geqslant \min\{0, p/\varepsilon\} + |p|/\varepsilon + \varepsilon > 0 \tag{3.16}$$

for every $\omega \in \Omega_1$. Hence,

$$\boldsymbol{P}(A > 1,\ B > 0) > 0. \tag{3.17}$$

Further, $AL \geqslant 0$ and so $\boldsymbol{P}(B + AL \geqslant 0) \geqslant \boldsymbol{P}(B > 0) > 0$. Thus $\bar{y} = \infty$.

4. Proofs

Proof of Lemma 1. The distribution of $\log A$ is not concentrated on a point since $w \in (0,\infty)$. By Theorem 7.1 and Corollary 7.1 of Barndorff-Nielsen (1978), the function c is strictly convex and continuously differentiable on $\mathring{\mathscr{D}}$. If $\boldsymbol{P}(A > 1)$ would equal zero then clearly, w would equal ∞. Thus $\boldsymbol{P}(A > 1) > 0$. Finally, $c(0) = c(w) = 0$. Because of strict convexity of c, we have $c(t) < 0$ for every $t \in (0,w)$ and thus $c'(w) > 0$. The proof of Lemma 1 is completed. □

We next state two lemmata which are needed in the proof of Theorem 2. For $m \in \mathbb{N}$, denote

$$\xi_m = B_m\mathbf{1}(B_m > 0) + A_m L_m \mathbf{1}(L_m > 0). \tag{4.1}$$

Let

$$Y_n' = 1 + \sum_{m=1}^{n} \Pi_{m-1}\xi_m \tag{4.2}$$

for $n \in \mathbb{N}$ where Π_{m-1} is as in (2.22) for $m \geqslant 2$ and, by convention, $\Pi_0 = 1$. Clearly, $Y_n' \geqslant Y_n$ for every $n \in \mathbb{N}$. Further, the sequence $\{Y_n' \mid n = 1, 2, \ldots\}$ is increasing and $Y_n' \geqslant 1$ for every $n \in \mathbb{N}$. For our purposes, Y_n' is sufficiently close to Y_n, and the above properties are useful in the subsequent proofs of upper bound results. In particular, it is possible to work with the moment generating function of $\log Y_n'$ and it is an increasing function for every $n \in \mathbb{N}$.

Define the function $h:\mathbb{R}\to\mathbb{R}\cup\{\pm\infty\}$ by

$$h(t)=\limsup_{n\to\infty} n^{-1}\log \boldsymbol{E}\{\mathrm{e}^{t\log Y_n'}\} \tag{4.3}$$

for $t\in\mathbb{R}$. Then h is an increasing function. Define further the function $g:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ by

$$g(t)=\begin{cases} 0 & \text{for } t\leqslant w,\\ c(t) & \text{for } t\in(w,t_0),\\ +\infty & \text{for } t\geqslant t_0. \end{cases} \tag{4.4}$$

Lemma 6. *Assume the conditions of Theorem* 2. *Then for every* $t\in\mathbb{R}$,

$$h(t)\leqslant g(t). \tag{4.5}$$

We also need a sample path result associated with the sequence $\{(A_n,B_n)\}$. Consider a fixed $x\in(x_0,\mu)$. Denote by $\lceil a\rceil$ the smallest integer $\geqslant a$. Let $\delta\in(0,x)$ and $\varepsilon'>0$. For $n\in\mathbb{N}$, define the continuous time process $z_n=\{z_n(\alpha)\,|\,0<\alpha<\infty\}$ by

$$z_n(\alpha)=(\log A_1+\cdots+\log A_{\lceil\alpha n\rceil})/n. \tag{4.6}$$

For $n\in\mathbb{N}$, denote

$$D_n=D_n(\delta,\varepsilon')=\left\{\omega\in\Omega\,|\sup_{0<\alpha\leqslant x-\delta}|z_n(\alpha)-\alpha/x|\leqslant\varepsilon'\right\} \tag{4.7}$$

and

$$E_n=E_n(\delta,\varepsilon')=\{\omega\in\Omega\,|\,|B_1|\leqslant\mathrm{e}^{\varepsilon' n},\ldots,|B_{\lceil(x-\delta)n\rceil}|\leqslant\mathrm{e}^{\varepsilon' n}\}. \tag{4.8}$$

The sample path described by the set D_n dominates the ruin event $\{T_M'\leqslant xM\}$ where T_M' is given by (1.3). This explains its importance in the subsequent proof. See Martin-Löf (1983) for the background.

Lemma 7. *Assume the conditions of Theorem* 2. *Let* $x\in(x_0,\mu)$. *Then for every* $\delta\in(0,x)$ *and* $\varepsilon'>0$,

$$\liminf_{n\to\infty} n^{-1}\log \boldsymbol{P}(D_n\cap E_n)\geqslant -xc^*(1/x). \tag{4.9}$$

Proof of Lemma 6. There is nothing to prove for $t\geqslant t_0$. Let $t\in(w,t_0)$. Assume first that $t>1$. By Minkowski's inequality, we have for $n\in\mathbb{N}$,

$$\begin{aligned}\boldsymbol{E}\{\mathrm{e}^{t\log Y_n'}\}^{1/t} &\leqslant 1+\sum_{m=1}^{n}\boldsymbol{E}\{(\Pi_{m-1}\xi_m)^t\}^{1/t}\\ &=1+\sum_{m=1}^{n}\mathrm{e}^{(m-1)c(t)/t}\boldsymbol{E}\{\xi_m^t\}^{1/t}\\ &=1+\mathrm{e}^{(n-1)c(t)/t}\boldsymbol{E}\{\xi_1^t\}^{1/t}\sum_{m=1}^{n}\mathrm{e}^{(m-n)c(t)/t}.\end{aligned} \tag{4.10}$$

For $t \in (w, t_0)$, we have $c(t) > 0$. Hence,

$$\sum_{m=1}^{n} \mathrm{e}^{(m-n)c(t)/t} \leqslant \sum_{m=0}^{\infty} \mathrm{e}^{-mc(t)/t} < \infty. \tag{4.11}$$

By (2.6), $t \in (w, t_0)$ and $w > 0$, $\boldsymbol{E}\{\xi_1^t\}$ is finite. It follows that

$$\boldsymbol{E}\{\mathrm{e}^{t \log Y_n'}\}^{1/t} \leqslant K\mathrm{e}^{(n-1)c(t)/t} \tag{4.12}$$

where $K \in (0, \infty)$ is a constant. This implies (4.5). Assume now that $t \leqslant 1$. Then, for every $y_1,\ y_2 \geqslant 0$, we have

$$(y_1 + y_2)^t \leqslant y_1^t + y_2^t. \tag{4.13}$$

Instead of Minkowski's inequality, we make use of this relation in (4.10) and obtain an upper bound which suffices for (4.12). Thus (4.5) holds for every $t > w$.

Let $t \leqslant w$. Recall that the function h is increasing on $(-\infty, t_0)$. We have already proved that $h(t) \leqslant c(t)$ for every $t \in (w, t_0)$. By Lemma 1, the function c is continuous on $(0, t_0)$. Hence,

$$h(t) \leqslant \limsup_{t \to w+} h(t) \leqslant \limsup_{t \to w+} c(t) = 0. \tag{4.14}$$

Thus (4.5) holds for every $t \in \mathbb{R}$. $\square$

Proof of Lemma 7. By Lemma 1, we may find $u \in \mathring{\mathscr{D}}$ such that $c'(u) = 1/x$. Then $u > w$, $c(u) > 0$ and $c^*(1/x) = u/x - c(u)$. See Rockafellar (1970, Theorem 23.5). Denote by P the distribution of (A, B) and by $Q = Q_u$ the distribution defined by

$$\mathrm{d}Q/\mathrm{d}P(y_1, y_2) = y_1^u \mathrm{e}^{-c(u)} \tag{4.15}$$

for $y_1, y_2 \in \mathbb{R}$. We will indicate in the sequel the distribution Q as a subscript in probabilities and expectations when a sequence of independent Q-distributed random vectors is considered.

Let $\varepsilon'' \in (0, \varepsilon')$. Then

$$\begin{aligned} \boldsymbol{P}(D_n(\delta,\varepsilon') \cap E_n(\delta,\varepsilon')) &\geqslant \boldsymbol{P}(D_n(\delta,\varepsilon'') \cap E_n(\delta,\varepsilon'')) \\ &= \boldsymbol{E}_Q\{\mathrm{e}^{-uz_n(x-\delta)n + \lceil (x-\delta)n \rceil c(u)} \mathbf{1}(D_n(\delta,\varepsilon'') \cap E_n(\delta,\varepsilon''))\} \\ &\geqslant \boldsymbol{E}_Q\{\mathrm{e}^{-u((x-\delta)/x+\varepsilon'')n + (x-\delta)nc(u)} \mathbf{1}(D_n(\delta,\varepsilon'') \cap E_n(\delta,\varepsilon''))\} \\ &= \mathrm{e}^{-(\varepsilon'' u + xc^*(1/x) - \delta c^*(1/x))n} \boldsymbol{P}_Q(D_n(\delta,\varepsilon'') \cap E_n(\delta,\varepsilon'')). \end{aligned} \tag{4.16}$$

Clearly,

$$\boldsymbol{E}_Q\{A^t\} = \mathrm{e}^{-c(u)} \boldsymbol{E}\{A^{t+u}\} < \infty \tag{4.17}$$

for every t in a neighbourhood of the origin. Consequently,

$$\begin{aligned} \boldsymbol{E}_Q\{\log A\} &= \partial/\partial t(\log \boldsymbol{E}_Q\{A^t\})_{|t=0} \\ &= \partial/\partial t(\log \boldsymbol{E}\{A^t\})_{|t=u} = c'(u) = 1/x. \end{aligned} \tag{4.18}$$

It follows for example from Theorem 6 of Nyrhinen (1995) that

$$\lim_{n\to\infty} \boldsymbol{P}_Q\left(\sup_{0<\alpha\leqslant x-\delta} |z_n(\alpha) - \alpha/x| \leqslant \varepsilon''\right) = 1. \tag{4.19}$$

Further,

$$\boldsymbol{E}_Q\{|B|^t\} = \mathrm{e}^{-c(u)}\boldsymbol{E}\{A^u|B|^t\} \tag{4.20}$$

for every $t > 0$. By Hölder's inequality, there exists $t' > 0$ such that

$$\boldsymbol{E}_Q\{|B|^{t'}\} < \infty. \tag{4.21}$$

By Chebycheff's inequality,

$$\begin{aligned}\boldsymbol{E}_Q\{|B|^{t'}\} &\geqslant \boldsymbol{E}_Q\{|B|^{t'}\mathbf{1}(|B| \geqslant \mathrm{e}^{\varepsilon'' n})\} \\ &\geqslant \mathrm{e}^{\varepsilon'' t' n}\boldsymbol{P}_Q(|B| \geqslant \mathrm{e}^{\varepsilon'' n}).\end{aligned} \tag{4.22}$$

Consequently,

$$\boldsymbol{P}_Q(E_n(\delta,\varepsilon'')) \geqslant 1 - \sum_{m=1}^{\lceil (x-\delta)n\rceil} \boldsymbol{P}_Q(|B_m| \geqslant \mathrm{e}^{\varepsilon'' n}) \geqslant 1 - \boldsymbol{E}_Q\{|B|^{t'}\}\lceil (x-\delta)n\rceil \mathrm{e}^{-\varepsilon'' t' n}. \tag{4.23}$$

By this, (4.19) and (4.21),

$$\lim_{n\to\infty} \boldsymbol{P}_Q(D_n(\delta,\varepsilon'') \cap E_n(\delta,\varepsilon'')) = 1. \tag{4.24}$$

By (4.16),

$$\liminf_{n\to\infty} n^{-1} \log \boldsymbol{P}(D_n(\delta,\varepsilon') \cap E_n(\delta,\varepsilon')) \geqslant -\varepsilon'' u - xc^*(1/x) + \delta c^*(1/x). \tag{4.25}$$

Since the function c^* is non-negative we obtain (4.9) by letting ε'' tend to zero. □

Proof of Theorem 2. Let $x \in (x_0, \mu)$. We begin by proving that

$$\limsup_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant x \log M) \leqslant -xc^*(1/x) \tag{4.26}$$

and

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant x \log M) \geqslant -xc^*(1/x). \tag{4.27}$$

These results imply (2.12) for every $x \in (x_0, \mu)$.

Consider (4.26). Denote by $\lfloor a \rfloor$ the integer part of $a \geqslant 0$. Clearly,

$$\{T_M \leqslant x \log M\} \subseteq \{Y'_{\lfloor x \log M\rfloor} > M\}. \tag{4.28}$$

Let $g^*(v) = \sup\{tv - g(t) \mid t \in \mathbb{R}\}$ for $v \in \mathbb{R}$ where g is defined in (4.4). Then, for every closed set $F \subseteq \mathbb{R}$,

$$\limsup_{n\to\infty} n^{-1} \log \boldsymbol{P}(\log Y'_n/n \in F) \leqslant -\inf\{g^*(v) \mid v \in F\}. \tag{4.29}$$

This follows for compact sets F from Theorem 2.1 of de Acosta (1985) and from Lemma 6. It also follows from Lemma 6 that $h(t) < \infty$ for every t in a neighbourhood of the origin. Then, by Lemma 1.2.18 and the proof of part (a) of Theorem 2.3.6 of Dembo and Zeitouni (1993), upper bound (4.29) holds for every closed set F.

By (4.28) and (4.29),

$$\begin{aligned}
&\limsup_{M\to\infty}(\log M)^{-1}\log \boldsymbol{P}(T_M \leqslant x\log M)\\
&\quad\leqslant \limsup_{M\to\infty}(\log M)^{-1}\log \boldsymbol{P}(\log Y'_{\lfloor x\log M\rfloor}\geqslant \log M)\\
&\quad\leqslant x\limsup_{M\to\infty}\lfloor x\log M\rfloor^{-1}\log \boldsymbol{P}(\log Y'_{\lfloor x\log M\rfloor}/\lfloor x\log M\rfloor\geqslant 1/x)\\
&\quad\leqslant -x\inf\{g^*(v)\mid v\geqslant 1/x\}.
\end{aligned}\tag{4.30}$$

By Lemma 1, (2.7), (2.8) and $x\in(x_0,\mu)$, there exists $u=u_x\in(w,t_0)$ such that $c'(u)=1/x$. Then $c^*(1/x)=u/x-c(u)$. See Rockafellar (1970, Theorem 23.5). For $v\geqslant 1/x$, we have

$$\begin{aligned}
g^*(v) &\geqslant uv-g(u)=uv-c(u)\\
&\geqslant u/x-c(u)=c^*(1/x).
\end{aligned}\tag{4.31}$$

This and (4.30) imply (4.26).

Consider (4.27). We have $\boldsymbol{P}(A>1)>0$ by Lemma 1. By the Heine–Borel Theorem, there exists $b>1$ such that $\boldsymbol{P}(|A-b|<\varepsilon)>0$ for every $\varepsilon>0$. Hence, we may fix $v<0$, $b>1$ and $\varepsilon\in(0,b-1)$ such that

$$q=\boldsymbol{P}(B>v,|A-b|<\varepsilon)>0. \tag{4.32}$$

Since $\bar{y}=\infty$ we may also fix $k\in\mathbb{N}$ such that

$$r=\boldsymbol{P}(Y_k>-v/(b-1-\varepsilon)+1)>0. \tag{4.33}$$

Let $\delta\in(0,x)$, $\delta'\in(0,\delta)$ and $\varepsilon'=\frac{1}{5}(\delta-\delta')\log(b-\varepsilon)$. Recall the definitions of D_n and E_n from (4.7) and (4.8). For sufficiently large n, we have for every $\omega\in D_n\cap E_n$,

$$Y^0_{\lceil(x-\delta)n\rceil}\geqslant -\lceil(x-\delta)n\rceil e^{\varepsilon' n}e^{((x-\delta)/x+\varepsilon')n}>-e^{(1-\delta/x+3\varepsilon')n}, \tag{4.34}$$

where $Y^0_{\lceil(x-\delta)n\rceil}$ is defined by (2.21). For $n\in\mathbb{N}$, denote

$$\begin{aligned}
F_n&=F_n(\delta,\delta')\\
&=\{\omega\in\Omega\mid B_i>v,|A_i-b|<\varepsilon \text{ for } i=\lceil(x-\delta)n\rceil+1,\ldots,\lceil(x-\delta')n\rceil\}.
\end{aligned}\tag{4.35}$$

Then

$$\boldsymbol{P}(F_n)=q^{\lceil(x-\delta')n\rceil-\lceil(x-\delta)n\rceil}\geqslant q^{(\delta-\delta')n+1}, \tag{4.36}$$

where q is as in (4.32). For $\omega\in F_n$, we have

$$\begin{aligned}
Y^0_{\lceil(x-\delta')n\rceil}-Y^0_{\lceil(x-\delta)n\rceil}&=\Pi_{\lceil(x-\delta')n\rceil}\sum_{i=\lceil(x-\delta)n\rceil+1}^{\lceil(x-\delta')n\rceil}B_i/(A_i\cdots A_{\lceil(x-\delta')n\rceil})\\
&\geqslant \Pi_{\lceil(x-\delta')n\rceil}\sum_{i=1}^{\infty}v/(b-\varepsilon)^i=\Pi_{\lceil(x-\delta')n\rceil}v/(b-1-\varepsilon).
\end{aligned}\tag{4.37}$$

For $n \in \mathbb{N}$, denote

$$G_n = G_n(\delta') = \{\omega \in \Omega \mid \Pi^{-1}_{\lceil (x-\delta')n\rceil}(Y_{\lceil (x-\delta')n\rceil+k} - Y^0_{\lceil (x-\delta')n\rceil}) > -v/(b-1-\varepsilon)+1\}. \tag{4.38}$$

Obviously, $\boldsymbol{P}(G_n) = r$ where r is as in (4.33). By combining (4.34), (4.37) and (4.38), it is seen that for large n for every $\omega \in D_n \cap E_n \cap F_n \cap G_n$,

$$\begin{aligned} Y_{\lceil (x-\delta')n\rceil+k} &= (Y_{\lceil (x-\delta')n\rceil+k} - Y^0_{\lceil (x-\delta')n\rceil}) + (Y^0_{\lceil (x-\delta')n\rceil} - Y^0_{\lceil (x-\delta)n\rceil}) + Y^0_{\lceil (x-\delta)n\rceil} \\ &\geqslant \Pi_{\lceil (x-\delta')n\rceil} - \mathrm{e}^{(1-\delta/x+3\varepsilon')n}. \end{aligned} \tag{4.39}$$

For $\omega \in D_n \cap F_n$, we have for large n,

$$\begin{aligned} \Pi_{\lceil (x-\delta')n\rceil} &\geqslant \mathrm{e}^{z_n(x-\delta)n}(b-\varepsilon)^{(\delta-\delta')n-1} \\ &\geqslant \mathrm{e}^{(1-\delta/x-\varepsilon')n}(b-\varepsilon)^{(\delta-\delta')n-1} \\ &= \mathrm{e}^{(1-\delta/x+4\varepsilon')n}/(b-\varepsilon). \end{aligned} \tag{4.40}$$

It follows from (4.39) that for sufficiently large n,

$$Y_{\lceil (x-\delta')n\rceil+k} > \mathrm{e}^{(1-\delta/x)n} \tag{4.41}$$

for every $\omega \in D_n \cap E_n \cap F_n \cap G_n$. The events $D_n \cap E_n$, F_n and G_n are mutually independent and $\boldsymbol{P}(G_n) = r > 0$ for every n. By (4.9), (4.36) and (4.41),

$$\begin{aligned} \liminf_{n\to\infty} & \; n^{-1} \log \boldsymbol{P}(Y_{\lceil (x-\delta')n\rceil+k} > \mathrm{e}^{(1-\delta/x)n}) \\ & \geqslant -xc^*(1/x) + (\delta-\delta')\log q. \end{aligned} \tag{4.42}$$

Clearly, for sufficiently large M,

$$\{Y_{\lceil (x-\delta')\lceil \log M\rceil\rceil+k} > \mathrm{e}^{(1-\delta/x)\lceil \log M\rceil}\} \subseteq \{T_{M^{1-\delta/x}} \leqslant x\log M\}. \tag{4.43}$$

By letting δ' tend to δ in (4.42), we obtain

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_{M^{1-\delta/x}} \leqslant x\log M) \geqslant -xc^*(1/x). \tag{4.44}$$

Consequently,

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant (x/(1-\delta/x))\log M) \geqslant -xc^*(1/x)/(1-\delta/x). \tag{4.45}$$

By letting δ tend to zero, it is seen that for given $\varepsilon'' > 0$,

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant (1+\varepsilon'')x\log M) \geqslant -xc^*(1/x). \tag{4.46}$$

Denote $x'' = x/(1+\varepsilon'')$. For small $\varepsilon'' > 0$, we have $x'' > x_0$. Then result (4.46) holds when x is replaced by x'' and so

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant x\log M) \geqslant -x''c^*(1/x''). \tag{4.47}$$

The function R defined by (2.9) is continuous. We obtain (4.27) by letting ε'' tend to zero.

Let now $x \in [\mu, \infty)$. Then

$$\boldsymbol{P}(T_M \leqslant x\log M) \geqslant \boldsymbol{P}(T_M \leqslant y\log M) \tag{4.48}$$

for every $y \in (x_0, \mu)$. Since R is continuous at μ we obtain by (4.27) and (2.9),

$$\liminf_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M \leqslant x \log M) \geqslant -\lim_{y\to\mu-} yc^*(1/y) = -w. \tag{4.49}$$

To see that (2.12) holds for every $x \geqslant \mu$, it suffices to show that

$$\limsup_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M < \infty) \leqslant -w. \tag{4.50}$$

We follow the lines of the proof of Theorem 1 in Nyrhinen (1999b). Denote

$$\bar{Y}' = \sup\{Y_n' \mid n \in \mathbb{N}\} \tag{4.51}$$

where Y_n' is defined in (4.2). Then $\boldsymbol{P}(T_M < \infty) \leqslant \boldsymbol{P}(\bar{Y}' > M)$. Suppose first that $w > 1$. Let $t \in (1, w)$. Then, inequality (4.10) holds. Since $c(t)$ is now negative we have

$$\boldsymbol{E}\{\mathrm{e}^{t \log Y_n'}\}^{1/t} \leqslant 1 + \boldsymbol{E}\{\xi_1^t\}^{1/t} \sum_{m=1}^{\infty} \mathrm{e}^{(m-1)c(t)/t} < \infty. \tag{4.52}$$

By the monotone convergence theorem,

$$\boldsymbol{E}\{(\bar{Y}')^t\} = \lim_{n\to\infty} \boldsymbol{E}\{\mathrm{e}^{t \log Y_n'}\} < \infty. \tag{4.53}$$

By Chebycheff's inequality,

$$\begin{aligned} \boldsymbol{E}\{(\bar{Y}')^t\} &\geqslant \boldsymbol{E}\{(\bar{Y}'\mathbf{1}(\bar{Y}' > M))^t\} \\ &\geqslant M^t \boldsymbol{P}(\bar{Y}' > M) \geqslant M^t \boldsymbol{P}(T_M < \infty). \end{aligned} \tag{4.54}$$

By (4.53),

$$\limsup_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}(T_M < \infty) \leqslant -t. \tag{4.55}$$

This implies (4.50). If $w \in (0, 1]$ then we make use of (4.13) instead of Minkowski's inequality in (4.10) and obtain (4.50) as above.

Limit (2.13) follows directly from (4.49) and (4.50). The proof of Theorem 2 is completed. □

For the proof of Theorem 3, we need the following technical lemma.

Lemma 8. *Assume the conditions of Lemma* 1 *and that the probability in* (2.23) *equals zero for every* $k \geqslant 1$. *Further, assume that* $t_0 = \infty$ *and that*

$$\boldsymbol{E}\{|L|^t\} < \infty \tag{4.56}$$

for every $t > 0$. *Then for given* $t > 1$, *there exists a finite constant* $K = K(t)$ *such that*

$$\boldsymbol{E}\{(Y_n \mathbf{1}(Y_n > 0))^t\} \leqslant K n^{t+1} \tag{4.57}$$

for every $n \in \mathbb{N}$.

Proof. Let $t > 1$. Denote

$$s = \sup\{y \in \mathbb{R} \mid \boldsymbol{P}(Y_k^0/(\Pi_k - 1) > y,\ \Pi_k > 1) > 0 \text{ for some } k \geqslant 1\}. \tag{4.58}$$

We have $\boldsymbol{P}(A>1)>0$ by Lemma 1. Thus $s>-\infty$. If s would equal ∞ then, by independence, (2.23) would hold for some $k\geqslant 1$. Hence, $s\in(-\infty,\infty)$. It follows that

$$\boldsymbol{P}(B_{m+1}+A_{m+1}L_{m+1}>-s)=0 \tag{4.59}$$

for every $m\in\mathbb{N}$. Thus

$$\boldsymbol{E}\{(Y_1\mathbf{1}(Y_1>0))^t\}<\infty \tag{4.60}$$

and so (4.57) holds for $n=1$.

Let $n\geqslant 2$. For fixed $m\in\mathbb{N}$, we have by (4.58) and (4.59),

$$\begin{aligned}\boldsymbol{P}(Y_{m+1}>-s,\,\Pi_m>1)&=\boldsymbol{P}(Y_m^0+\Pi_m(B_{m+1}+A_{m+1}L_{m+1})>-s,\,\Pi_m>1)\\&\leqslant \boldsymbol{P}((\Pi_m-1)s+\Pi_m(B_{m+1}+A_{m+1}L_{m+1})>-s,\,\Pi_m>1)\\&=\boldsymbol{P}(B_{m+1}+A_{m+1}L_{m+1}>-s,\,\Pi_m>1)=0.\end{aligned} \tag{4.61}$$

Denote

$$\tau_n=\begin{cases}\max\{m\in\mathbb{N}\mid m\leqslant n-1,\,\Pi_m>1\}\\ 0\quad\text{if }\Pi_m\leqslant 1\ \text{ for } m=1,\ldots,n-1.\end{cases} \tag{4.62}$$

For $m=1,\ldots,n-2$, we have by (4.61),

$$\begin{aligned}&\boldsymbol{E}\{(Y_n\mathbf{1}(Y_n>0))^t\mathbf{1}(\tau_n=m)\}\\&\quad=\boldsymbol{E}\{((Y_n-Y_{m+1}+Y_{m+1})\mathbf{1}(Y_n>0,\tau_n=m))^t\}\\&\quad\leqslant\boldsymbol{E}\left\{\left(\left(-\Pi_{m+1}L_{m+1}+\sum_{i=m+2}^{n}\Pi_{i-1}B_i+\Pi_{n-1}A_nL_n-s\right)\mathbf{1}(Y_n>0,\tau_n=m)\right)^t\right\}\\&\quad\leqslant\boldsymbol{E}\left\{\left(|L_{m+1}|+\sum_{i=m+2}^{n}|B_i|+A_n|L_n|+|s|\right)^t\right\}.\end{aligned} \tag{4.63}$$

For $m=n-1$, we have by (4.61),

$$\boldsymbol{E}\{(Y_n\mathbf{1}(Y_n>0))^t\mathbf{1}(\tau_n=m)\}\leqslant|s|^t \tag{4.64}$$

and for $m=0$,

$$\boldsymbol{E}\{(Y_n\mathbf{1}(Y_n>0))^t\mathbf{1}(\tau_n=m)\}\leqslant\boldsymbol{E}\{(|B_1|+\cdots+|B_n|+A_n|L_n|)^t\}. \tag{4.65}$$

Recall (4.56) and that $t_0=\infty$. It follows from Minkowski's inequality and from (4.63), (4.64) and (4.65) that for every $n\geqslant 2$ and $m=0,1,\ldots,n-1$,

$$\boldsymbol{E}\{(Y_n\mathbf{1}(Y_n>0))^t\mathbf{1}(\tau_n=m)\}\leqslant K'n^t \tag{4.66}$$

where $K'=K'(t)$ is a finite constant, independent of n and m. Consequently,

$$\boldsymbol{E}\{(Y_n\mathbf{1}(Y_n>0))^t\}\leqslant K'n^{t+1} \tag{4.67}$$

for every $n\geqslant 2$. This and (4.60) imply (4.57). □

Proof of Theorem 3. Let $k \geqslant 1$ be such that (2.23) holds. We have to show that $\bar{y} = \infty$. Fix $\gamma > 0$ such that

$$\boldsymbol{P}(B_{k+1} + A_{k+1}L_{k+1} + Y_k^0/(\Pi_k - 1) > \gamma,\ 1 + \gamma \leqslant \Pi_k \leqslant 1/\gamma,\ |Y_k^0| \leqslant 1/\gamma) > 0. \quad (4.68)$$

An application of the Heine–Borel Theorem shows that we may fix $b \geqslant 1 + \gamma$ and $v \in [-1/\gamma, 1/\gamma]$ such that

$$\boldsymbol{P}(B_{k+1} + A_{k+1}L_{k+1} + Y_k^0/(\Pi_k - 1) > \gamma,\ |\Pi_k - b| < \varepsilon,\ |Y_k^0 - v| < \varepsilon) > 0 \quad (4.69)$$

for every $\varepsilon \in (0, b - 1)$.

Suppose first that $v > 0$. For $m \in \mathbb{N}$, denote

$$H_m = H_m(\varepsilon) = \{\omega \in \Omega \mid B_m + A_m L_m > \gamma - (v + \varepsilon)/(b - 1 - \varepsilon)\}. \quad (4.70)$$

By (4.69), $\boldsymbol{P}(H_m) = \boldsymbol{P}(H_1) > 0$ for every $m \in \mathbb{N}$. For $n \in \mathbb{N}$, denote

$$A'_n = A_{(n-1)k+1} \cdots A_{nk} \quad (4.71)$$

and

$$B'_n = B_{(n-1)k+1} + A_{(n-1)k+1}B_{(n-1)k+2} + \cdots + A_{(n-1)k+1} \cdots A_{nk-1}B_{nk}. \quad (4.72)$$

Then $(A'_1, B'_1) = (\Pi_k, Y_k^0)$ and the random vectors $(A'_1, B'_1), (A'_2, B'_2), \ldots$ are independent and identically distributed. Denote

$$I_n = I_n(\varepsilon) = \{\omega \in \Omega \mid |A'_i - b| < \varepsilon,\ |B'_i - v| < \varepsilon \quad \text{for } i = 1, \ldots, n\}.$$

By (4.69), $\boldsymbol{P}(I_n) = \boldsymbol{P}(I_1)^n > 0$.

Let ε be such that $v - \varepsilon > 0$. For $\omega \in I_n$, we have

$$B'_j/(A'_j \cdots A'_n) \geqslant (v - \varepsilon)/(b + \varepsilon)^{n-j+1} \quad (4.73)$$

for $j = 1, \ldots n$. Thus, for $\omega \in H_{nk+1} \cap I_n$,

$$\begin{aligned} Y_{nk+1} &= B'_1 + A'_1B'_2 + \cdots + A'_1 \cdots A'_{n-1}B'_n + A'_1 \cdots A'_n(B_{nk+1} + A_{nk+1}L_{nk+1}) \\ &\geqslant A'_1 \cdots A'_n \left(\sum_{j=1}^{n} (v - \varepsilon)/(b + \varepsilon)^j + B_{nk+1} + A_{nk+1}L_{nk+1} \right) \\ &\geqslant A'_1 \cdots A'_n((v - \varepsilon)/(b - 1 + \varepsilon) \\ &\quad - \sum_{j=n+1}^{\infty} (v - \varepsilon)/(b + \varepsilon)^j + \gamma - (v + \varepsilon)/(b - 1 - \varepsilon)). \end{aligned} \quad (4.74)$$

Choose small ε and large n such that

$$\left| \sum_{j=n+1}^{\infty} (v - \varepsilon)/(b + \varepsilon)^j \right| \leqslant \gamma/4 \quad (4.75)$$

and

$$|(v - \varepsilon)/(b - 1 + \varepsilon) - (v + \varepsilon)/(b - 1 - \varepsilon)| \leqslant \gamma/4. \quad (4.76)$$

Then

$$Y_{nk+1} \geqslant (b - \varepsilon)^n \gamma/2 \quad (4.77)$$

for every $\omega \in H_{nk+1} \cap I_n$. The events H_{nk+1} and I_n are independent. Thus

$$\boldsymbol{P}(H_{nk+1} \cap I_n) = \boldsymbol{P}(H_1)\boldsymbol{P}(I_1)^n > 0. \tag{4.78}$$

Since $b - \varepsilon > 1$ we conclude that $\bar{y} = \infty$.

If $v = 0$ then estimate (4.73) is not available. Instead of that, we make use of the estimate

$$B'_j/(A'_j \cdots A'_n) \geqslant (v - \varepsilon)/(b - \varepsilon)^{n-j+1}$$

and proceed as in the case $v > 0$. The same estimate is used for negative v. Additionally, in this case it is not guaranteed that $\boldsymbol{P}(H_m)$ is positive. Instead of H_m, we consider the set

$$H'_m = \{\omega \in \Omega \mid B_m + A_m L_m > \gamma - (v + \varepsilon)/(b - 1 + \varepsilon)\}. \tag{4.79}$$

Then $\boldsymbol{P}(H'_m) > 0$ and it is seen as above that $\bar{y} = \infty$.

Assume now that

$$\boldsymbol{P}(B + AL + Y_k^0/(\Pi_k - 1) > 0, \Pi_k > 1) = 0 \tag{4.80}$$

for $k = 1, 2, \ldots$. We have to prove that $\bar{y}$ is finite. Denote by P' the distribution of (A, B, L). For $v > 0$, define the distribution $Q' = Q'_v$ by

$$\mathrm{d}Q'/\mathrm{d}P'(y_1, y_2, y_3) = \kappa \prod_{i=1}^{3} [\mathbf{1}(|y_i| \leqslant v) + \mathrm{e}^{-|y_i|}\mathbf{1}(|y_i| > v)] \tag{4.81}$$

for $y_1, y_2, y_3 \in \mathbb{R}$ where $\kappa \in [1, \infty)$ has been chosen such that $Q'(\mathbb{R}^3) = 1$. Associated with the process $\{Y_n\}$ with Q' as the distribution of (A, B, L), the parameter w is finite and positive for large v. Under the distribution Q', the expectations of the variables A^t, $|B|^t$ and $|L|^t$ are finite for every $t > 0$. By Hölder's inequality, the expectation of $(A|L|)^t$ is also finite for every $t > 0$. The distributions P' and Q' are mutually absolutely continuous and so (4.80) is preserved in this change of measure. Also $\bar{y}$ remains unchanged.

We indicate in the sequel the distribution Q' as a subscript in probabilities and expectations when a sequence of independent Q'-distributed random vectors is considered. By the above discussion, we may assume that under the distribution Q', all the conditions of Lemma 8 are satisfied. Let $x \in (x_0, \infty)$ where x_0 is defined by the distribution Q'. We conclude by Lemma 8 and by Chebycheff's inequality that for every $n \leqslant \lceil x \log M \rceil$,

$$\begin{aligned}\boldsymbol{P}_{Q'}(Y_n > M) &\leqslant M^{-t}\boldsymbol{E}_{Q'}\{(Y_n\mathbf{1}(Y_n > 0))^t\} \\ &\leqslant KM^{-t}n^{t+1} \leqslant KM^{-t}\lceil x \log M \rceil^{t+1}.\end{aligned} \tag{4.82}$$

Hence,

$$\begin{aligned}\boldsymbol{P}_{Q'}(T_M \leqslant x \log M) &\leqslant \sum_{n=1}^{\lceil x \log M \rceil} \boldsymbol{P}_{Q'}(Y_n > M) \\ &\leqslant KM^{-t}\lceil x \log M \rceil^{t+2}.\end{aligned} \tag{4.83}$$

Consequently,

$$\limsup_{M\to\infty} (\log M)^{-1} \log \boldsymbol{P}_{Q'}(T_M \leqslant x \log M) \leqslant -t. \tag{4.84}$$

If $\bar{y}$ would equal ∞ then by Theorem 2, limit (2.12) would hold with $R(x) < \infty$. Since t in (4.84) is arbitrary we conclude that $\bar{y}$ is finite. □

Proof of Theorem 4. Result (3.5) follows from (3.4) since $\boldsymbol{P}(T_M \leqslant n)$ tends to $\boldsymbol{P}(T_M < \infty)$ and $\boldsymbol{P}(T_M^{\mathrm{c}} \leqslant n)$ to $\boldsymbol{P}(T_M^{\mathrm{c}} < \infty)$ when n tends to infinity. Consider (3.4). For $n \in \mathbb{N}$, denote

$$\bar{Y}_n = \sup\{Y_m \mid m = 1, 2, \ldots, n\} \tag{4.85}$$

and

$$\bar{Y}_n^{\mathrm{c}} = \sup\{Y_\tau^{\mathrm{c}} \mid \tau \in [0, n]\}. \tag{4.86}$$

Then

$$\boldsymbol{P}(T_M \leqslant n) = \boldsymbol{P}(\bar{Y}_n > M) \tag{4.87}$$

and

$$\boldsymbol{P}(T_M^{\mathrm{c}} \leqslant n) = \boldsymbol{P}(\bar{Y}_n^{\mathrm{c}} > M). \tag{4.88}$$

The choice of the vector (A, B, L) implies that $\bar{Y}_1$ and $\bar{Y}_1^{\mathrm{c}}$ are identically distributed. Let $n \geqslant 2$. Similarly to (2.18), it is seen that

$$\bar{Y}_n =_L B + A \max\{L, \bar{Y}_{n-1}\}, \tag{4.89}$$

where $\bar{Y}_{n-1}$ is independent of A, B and L. For almost all $\omega \in \Omega$, we have

$$\begin{aligned}
\bar{Y}_n^{\mathrm{c}} &= \max\{\sup\{Y_\tau^{\mathrm{c}} \mid \tau \in [0,1]\}, \sup\{Y_\tau^{\mathrm{c}} \mid \tau \in [1,n]\}\} \\
&= Y_1^{\mathrm{c}} + \max\left\{\sup\{Y_\tau^{\mathrm{c}} - Y_1^{\mathrm{c}} \mid \tau \in [0,1]\}, \sup\left\{\int_1^\tau A_{s-}^{\mathrm{c}}\, \mathrm{d}B_s^{\mathrm{c}} \mid \tau \in [1,n]\right\}\right\} \\
&= Y_1^{\mathrm{c}} + A_1^{\mathrm{c}} \max\left\{(A_1^{\mathrm{c}})^{-1} \sup\{Y_\tau^{\mathrm{c}} - Y_1^{\mathrm{c}} \mid \tau \in [0,1]\},\right. \\
&\qquad \left.\sup\left\{\int_1^\tau A_{s-}^{\mathrm{c}}/A_1^{\mathrm{c}}\, \mathrm{d}B_s^{\mathrm{c}} \mid \tau \in [1,n]\right\}\right\}.
\end{aligned} \tag{4.90}$$

Since $A_1^{\mathrm{c}} = A_{1-}^{\mathrm{c}}$ almost surely and $\{(\log A_\tau^{\mathrm{c}}, B_\tau^{\mathrm{c}})\}$ is a Lévy process it follows that

$$\bar{Y}_n^{\mathrm{c}} =_L B + A \max\{L, \bar{Y}_{n-1}^{\mathrm{c}}\} \tag{4.91}$$

where $\bar{Y}_{n-1}^{\mathrm{c}}$ is independent of A, B and L. By (4.89) and (4.91), we conclude by induction that $\bar{Y}_n$ and $\bar{Y}_n^{\mathrm{c}}$ are identically distributed for every $n \in \mathbb{N}$. We obtain (3.4) by (4.87) and (4.88). □

Proof of Corollary 5. Result (3.7) follows directly from (2.13) and (3.5) and result (3.8) from (2.17) and (3.5). Consider (3.6). Fix $x > x_0$. Let $\varepsilon > 0$ be such that $x - \varepsilon > x_0$. By (3.4), we have for sufficiently large M,

$$\begin{aligned}
\boldsymbol{P}(T_M^{\mathrm{c}} \leqslant x \log M) &\geqslant \boldsymbol{P}(T_M^{\mathrm{c}} \leqslant \lfloor x \log M \rfloor) \\
&= \boldsymbol{P}(T_M \leqslant \lfloor x \log M \rfloor) \geqslant \boldsymbol{P}(T_M \leqslant (x - \varepsilon) \log M).
\end{aligned} \tag{4.92}$$

By Theorem 2,

$$\liminf_{M\to\infty} (\log M)^{-1}\log \boldsymbol{P}(T_M^{\mathrm{c}} \leqslant x\log M) \geqslant -R(x-\varepsilon). \tag{4.93}$$

Since R is continuous on (x_0,∞) we conclude by letting ε tend to zero that

$$\liminf_{M\to\infty} (\log M)^{-1}\log \boldsymbol{P}(T_M^{\mathrm{c}} \leqslant x\log M) \geqslant -R(x). \tag{4.94}$$

The proof of the reverse inequality is similar. Thus (3.6) holds. □

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